Number Theory 19

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Q) Let $n, p > 1$ be positive integers and p be a prime. If $n \mid p-1$ and $p \mid n^3-1$, prove that $4p-3$ is a perfect square.

Ans:- $p \mid n^3-1 \Rightarrow p \mid (n-1)(n^2+n+1)$

as $n \mid (p-1) \Rightarrow n \leq p-1$

$\Rightarrow p \mid (n^2+n+1)$

Let $p = nk_1+1$

$\Rightarrow p = (nk_1+1) \mid (n^2+n+1)$

$\Rightarrow nk_1+1 \leq n^2+n+1$

$\Rightarrow k_1 \leq n+1$

$\Rightarrow k_1 - n \leq 1$

Also, $k_1n+1 \mid (k_1n^2+k_1n+k_1)$

$\Rightarrow k_1n+1 \mid (k_1n^2+k_1n+k_1 - n(k_1n+1))$

$\Rightarrow k_1n+1 \mid (k_1n+k_1-n)$

$\Rightarrow k_1n+1 \leq k_1n+(k_1-n)$

$\Rightarrow k_1 - n \geq 1$

$\Rightarrow k_1 - n = 1$

$\Rightarrow k_1 = n+1$

$\Rightarrow p = n(n+1)+1 = n^2+n+1$

$\Rightarrow 4p-3 = 4n^2+4n+4-3 = 4n^2+4n+1 = (2n+1)^2$ //

Q) Evaluate $\left\lfloor \frac{2^0}{3} \right\rfloor + \left\lfloor \frac{2^1}{3} \right\rfloor + \left\lfloor \frac{2^2}{3} \right\rfloor + \cdots + \left\lfloor \frac{2^{1000}}{3} \right\rfloor$

In mod 3,

Ans:— $2^0 \equiv 1, \quad 2^1 = 2, \; 2^2 = 1, \; 2^3 = 2, \; 2^4 = 1, \ldots$

$\Rightarrow \left\lfloor \frac{2^0}{3} \right\rfloor + \left\lfloor \frac{2^1}{3} \right\rfloor + \left\lfloor \frac{2^2}{3} \right\rfloor + \left\lfloor \frac{2^4}{3} \right\rfloor + \cdots + \left\lfloor \frac{2^{1000}}{3} \right\rfloor$

$= \frac{2^0-1}{3} + \frac{2^1-2}{3} + \frac{2^2-1}{3} + \frac{2^4-2}{3} + \cdots + \frac{2^{1000}-1}{3}$

$= 0 + \frac{1}{3}\left[\sum_{i=1}^{1000} 2^i - 500 \times (2+1) \right]$

$= \frac{1}{3} \sum_{i=1}^{1000} 2^i - 500$

$= \frac{1}{3} (2^{1001} - 2) - 500$

$\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$

$\sum_{i=1}^{n} 2^i = 2^{n+1} - 2$

$n \quad \cdot \quad 2^{n+1} - 1$

$$= \frac{1}{3} \sum_{i=1} 2 \quad -$$

$$= \frac{1}{3}\left(2^{1001}-2\right) - 500$$

$$\sum_{i=1} 2 - 2$$

$$\sum_{i=0}^{n} a^i = \frac{a^{n+1}-1}{a-1}$$

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$$

for large n it tends to $\log n$

Q) Find the remainder when, $2024^{2023^{2022^{\cdot^{\cdot^{2^1}}}}} + 2025^{2024^{2017^{\cdot^{\cdot^{5^1}}}}}$

is divided by 19.

Ans:— (to be done later)

Q) Find positive reals a, b, c such that, $\sqrt{\frac{a}{b+c}} + \sqrt{\frac{b}{c+a}} + \sqrt{\frac{c}{a+b}} = 2$.

Ans:— Let $a \geq b \geq c$

$$\frac{\sqrt{a}}{\sqrt{b+c}} + \frac{\sqrt{b}}{\sqrt{c+a}} + \frac{\sqrt{c}}{\sqrt{a+b}} \geq 3\sqrt[3]{\frac{\sqrt{abc}}{\sqrt{a+b}\sqrt{b+c}\sqrt{c+a}}}$$

For equality,

$$\frac{\sqrt{a}}{\sqrt{b+c}} = \frac{\sqrt{b}}{\sqrt{c+a}} = \frac{\sqrt{c}}{\sqrt{a+b}}$$

$$\Rightarrow \frac{a}{b+c} = \frac{b}{c+a} = \frac{c}{a+b}$$

$\Rightarrow ac + a^2 = b^2 + bc$

$\Rightarrow a^2 - b^2 = bc - ac$

$\Rightarrow (a-b)(a+b) = c(b-a)$

If $a \neq b$ $\Rightarrow a+b = -c \Rightarrow a+b+c = 0$ × not possible

If $a = b$, then,

$$\frac{b}{c+a} = \frac{c}{a+b}$$

$\Rightarrow \frac{a}{c+a} = \frac{c}{2a} \Rightarrow 2a^2 = c^2 + a^2$

$\Rightarrow a^2 = c^2$

$\Rightarrow a = c$

$\Rightarrow a = b = c$

$\Rightarrow a = b = c$ is must for equality

Then $\frac{\sqrt{a}}{} + \frac{\sqrt{b}}{} + \frac{\sqrt{c}}{} \geq 3\sqrt[3]{\frac{\sqrt{a^3}}{\sqrt{2^3 a^3}}} = 3\frac{\sqrt{a}}{\sqrt{2a}} = \frac{3}{\sqrt{2}} > 2$

— ,

Then, $\frac{\sqrt{a}}{\sqrt{b+c}} + \frac{\sqrt{b}}{\sqrt{c+a}} + \frac{\sqrt{c}}{\sqrt{a+b}} \geq 3\sqrt[3]{\frac{\sqrt{a^3}}{\sqrt{2^3a^3}}} = 3\frac{\sqrt{a}}{\sqrt{2a}} = \frac{3}{\sqrt{2}} > 2$

$\Rightarrow$ No solution exists